A fancy way to obtain the binary digits of $759250125\sqrt{2}$

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1 Introduction.

In the present note we give some "easily-stated" recurrences of a special type that generate the binary digits for some "complicated" real numbers, such as the one in the title. The binary digits of any real number $t = (d_1.d_2d_3\cdots)_2$ with $1 \le t < 2$ can be calculated by the formula

$$d_n = \lfloor t2^{n-1} \rfloor - 2\lfloor t2^{n-2} \rfloor, \qquad n \ge 1.$$

We here show a somewhat unexpected, "fancy" way to obtain the digits of some special multiples of $\sqrt{2}$, where it is possible to hide this calculation.

This note is structured as follows. In Section 2 we first recall what is known about the Graham–Pollak sequence and its variants, which serve as the motivating examples for the definition of the so-called Graham–Pollak pairs in Section 3. We then give the general theorem (Theorem 3.2), which in particular provides a new extension of the original result due to Graham and Pollak. Finally, we give Corollary 3.3 as one surprising example of the general phenomenon.

2 The Graham–Pollak sequence.

As usual, denote by $\lfloor x \rfloor$ the greatest integer less than or equal to $x \in \mathbb{R}$, and by $\{x\}$ the fractional part of x. Define the sequence $(u_n)_{n\ge1}$ by the recurrence

$$u_1 = 1, \qquad u_{n+1} = \left\lfloor \sqrt{2}\left(u_n + \frac{1}{2}\right) \right\rfloor, \quad n \ge 1. \tag{1}$$

This sequence, which is also known as the *Graham–Pollak sequence*, first appeared in a proceedings paper of F. K. Hwang and S. Lin [8] in the framework of Ford and Johnson's sorting algorithm [4]. For the reader interested in the background of the algorithm, an updated exposition can be found in the third volume of D. E. Knuth's *The Art of Computer Programming* [9, Ch. 5.3.1, pp. 188]. The sequence (1) was first investigated from a purely mathematical point of view by R. L. Graham and H. O. Pollak [6]. They found the particularly intriguing fact that

$$d_n = u_{2n+1} - 2u_{2n-1} \tag{2}$$

gives the nth binary digit of $\sqrt{2} = (1.011010100\ldots)_2$.

This fact puzzled several authors since then, and it has often been included as a fun exercise in articles and books mostly on combinatorial number theory. We mention, for instance, P. Erdős and R. L. Graham [3, p. 96], R. Guy [7, Ex. 30], R. L. Graham, D. E. Knuth, and O. Patashnik [5,

Ex. 3.46]. More recent references are J.-P. Allouche and J. Shallit [1, Ex. 45, p. 116] and J. Borwein and D. Bailey [2, p. 62–63]. N. J. A. Sloane's online encyclopedia of integer sequences [12] gives eight sequences which are connected to the Graham–Pollak sequence (1), namely, A091522, A091523, A091524, A091525, A100671, A100673, A001521, and A004539.

Recently [13, 14], the present author found vast extensions of the Graham–Pollak sequence to parametric families of recurrences, where the initial value $u_1 = 1$ is replaced by $u_1 = m$ and the $\sqrt{2}$ in the recurrence is accordingly changed. However, the sequence is still wrapped in considerable mystery. Indeed, if we do not alter the $\sqrt{2}$ in the recurrence, but on the other hand, allow only the $1/2$ to vary (if n is odd), some quite strange things happen: we get the digits of various different multiples of $\sqrt{2}$, whose digits are seemingly unrelated. We point out that if we let the $1/2$ vary for n even instead (cf. [14, Theorem 3.3]), such effects cannot be observed.

3 Main Result.

In this note we are concerned with the following type of recurrences.

Definition 3.1. Let $\varepsilon \in \mathbb{R}$ and define the sequence $(v_n)_{n\geq 1}$ by

$$v_1 = 1, \qquad v_{n+1} = \begin{cases} \lfloor \sqrt{2}\,(v_n + \varepsilon) \rfloor, & \text{if } n \text{ is odd;} \\ \lfloor \sqrt{2}\,(v_n + \frac{1}{2}) \rfloor, & \text{if } n \text{ is even.} \end{cases}$$

We call (ε, t) a *Graham–Pollak pair* if the sequence

$$d_n = v_{2n+1} - 2v_{2n-1}, \qquad n \geq 1,$$

represents the binary digits of t; that is, $t = (d_1.d_2d_3\ldots)_2$.

Note that $(1/2, \sqrt{2})$ is a Graham–Pollak pair according to the original result about the sequence (1). Our main result is as follows:

Theorem 3.2. *A list of Graham–Pollak pairs is given by*

$$\{(\varepsilon_i, t_i): \quad 1 \leq i \leq 8\},$$

where

$$\begin{aligned}
1 - \frac{\sqrt{2}}{2} &\leq \varepsilon_1 < \sqrt{2} - 1, & t_1 &= \sqrt{2} - 1, \\
\sqrt{2} - 1 &\leq \varepsilon_2 < \frac{19}{2}\sqrt{2} - 13, & t_2 &= \frac{11}{8}\sqrt{2} - \frac{5}{8}, \\
\frac{19}{2}\sqrt{2} - 13 &\leq \varepsilon_3 < \frac{77}{2}\sqrt{2} - 54, & t_3 &= \frac{45}{32}\sqrt{2} - \frac{19}{32}, \\
\frac{77}{2}\sqrt{2} - 54 &\leq \varepsilon_4 < \frac{309}{2}\sqrt{2} - 218, & t_4 &= \frac{181}{128}\sqrt{2} - \frac{75}{128}, \\
\frac{309}{2}\sqrt{2} - 218 &\leq \varepsilon_5 < \frac{1296121037}{2}\sqrt{2} - 916495974, & t_5 &= \sqrt{2}, \\
\frac{1296121037}{2}\sqrt{2} - 916495974 &\leq \varepsilon_6 < \frac{79109}{2}\sqrt{2} - 55938, & t_6 &= \frac{759250125}{536870912}\sqrt{2} - \frac{314491699}{536870912}, \\
\frac{79109}{2}\sqrt{2} - 55938 &\leq \varepsilon_7 < \frac{5}{2}\sqrt{2} - 3, & t_7 &= \frac{46341}{32768}\sqrt{2} - \frac{19195}{32768}, \\
\frac{5}{2}\sqrt{2} - 3 &\leq \varepsilon_8 < \frac{\sqrt{2}}{2}, & t_8 &= \frac{3}{2}\sqrt{2} - \frac{1}{2}.
\end{aligned}$$

We first comment on a few aspects of the theorem.

(a) A surprising feature of Theorem 3.2 is that as ε varies continuously, the output makes discrete jumps among multiples of $\sqrt{2}$. Figure 1 illustrates the various intervals for ε and the corresponding numbers t appearing in Theorem 3.2.

(b) The binary digits of $\sqrt{2}$ are obtained for any choice of ε in the interval

$$[0.4959953\cdots, 0.5012400\cdots).$$

This slightly generalizes the original result of Graham and Pollak with $\varepsilon = 1/2$.

(c) There may well exist Graham–Pollak pairs besides those given in Theorem 3.2. However, it is easily checked with a computer that the range for admissible values of ε cannot be too large. For example, for $\varepsilon = 0.2928$ we get $d_{3067} = -1$, and for $\varepsilon = 0.7073$ we have $d_{2293} = 2$. Similar phenomena hold outside these bounds where the values $d_n = -1$ and $d_n = 2$ are already obtained for smaller indices n. Therefore, possible new pairs can only arise in a very small neighborhood of $\varepsilon = 1 - \frac{\sqrt{2}}{2}$ or $\varepsilon = \frac{\sqrt{2}}{2}$.

(d) There is also a connection to normal numbers, which shows that a characterization result for Graham–Pollak pairs is very difficult to obtain. Suppose that there exists $c_1 \in \mathbb{R}$ such that

$$\{\sqrt{2}\, 2^{k-1}\} \le c_1 < 1, \qquad \text{for all } k \ge 1. \tag{3}$$

Then – according to (8) below – the interval given for ε_1 can be enlarged to $c_1\left(1 - \frac{\sqrt{2}}{2}\right) \le \varepsilon_1 < \sqrt{2} - 1$. Similarly, if there is $c_2 \in \mathbb{R}$ such that

$$\{3\sqrt{2}\, 2^{k-2}\} \ge c_2 > 0, \qquad \text{for all } k \ge 1, \tag{4}$$

then the interval for ε_8 can be enlarged to $\frac{5}{2}\sqrt{2} - 3 \le \varepsilon_8 < \frac{\sqrt{2}}{2} + c_2\left(1 - \frac{\sqrt{2}}{2}\right)$. The inequality (3) implies that $\sqrt{2}$ is not normal in base two, and (4) implies that $3\sqrt{2}$ is not normal [10, Ch. 1.8].

We conclude with a surprising example, which follows from Theorem 3.2 by the (rather plain) observation that the number $1 - \frac{\pi^2}{e^3} = 0.5086213\ldots$ lies in the interval given for ε_6.

Corollary 3.3. *Define the sequence $(w_n)_{n\ge1}$ by*

$$w_1 = 1, \qquad w_{n+1} = \begin{cases} \lfloor \sqrt{2}\,(w_n + 1 - \frac{\pi^2}{e^3}) \rfloor, & \text{if } n \text{ is odd;} \\ \lfloor \sqrt{2}\,(w_n + \frac{1}{2}) \rfloor, & \text{if } n \text{ is even.} \end{cases}$$

Then for $n \ge 31$, $w_{2n+1} - 2w_{2n-1}$ is the $(n+1)$th binary digit of $759250125\sqrt{2}$.

4 Proof of Theorem 3.2.

First, let $i \in I := \{1, 2, \ldots, 8\} \setminus \{5\}$ and consider the pairs (ε_i, t_i) in the statement of Theorem 3.2. Put

$$t_i = (\alpha_i\sqrt{2} - \beta_i) \cdot 2^{-l_i}$$

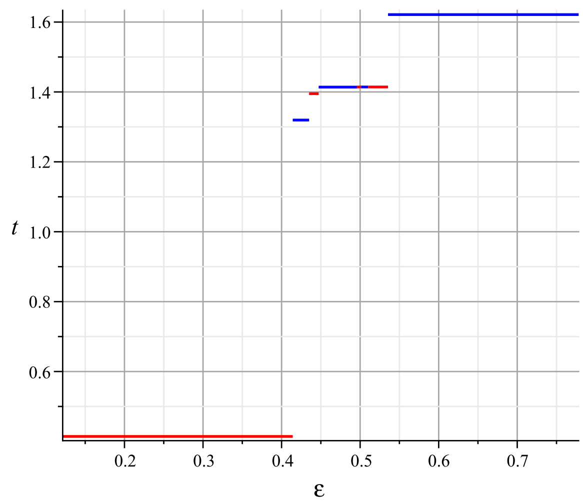

Figure 1: The sets $\{(\varepsilon_i, t_i)\}$ for $i = 1, \ldots, 8$.

with $\alpha_i, \beta_i, l_i \in \mathbb{Z}$ and $(\alpha_i, 2) = 1$. It is easy to verify that $\alpha_i + \beta_i = 2^{l_i+1}$ for $i \in I$. Furthermore, let $\xi_{1,i}$ and $\xi_{2,i}$ be the endpoints of the associated interval for ε_i. We shall prove that for $\xi_{1,i} \le \varepsilon_i < \xi_{2,i}$ and $k \ge l_i + 2$ we have

$$v_{2k} = \lfloor t_i 2^{k-2} \rfloor + \gamma_i 2^{k-l_i-2}, \tag{5}$$

$$v_{2k+1} = \lfloor t_i 2^{k-1} \rfloor + 2^k, \tag{6}$$

where $\gamma_i = 2\alpha_i + \beta_i$. This then implies that for $k \ge l_i + 3$,

$$v_{2k+1} - 2v_{2k-1} = \lfloor t_i 2^{k-1} \rfloor - 2\lfloor t_i 2^{k-2} \rfloor,$$

which is the kth binary digit of t_i. In the final step we then show that formula (6) indeed holds true for $0 \le k \le l_i + 1$, which completes the proof.

We first use induction to prove that if (5) holds for $k = l_i + 2$, then (5) and (6) hold for $k \ge l_i + 2$. Assume the validity of (5). We have to show that $\left\lfloor \sqrt{2}\left(v_{2k} + \frac{1}{2}\right) \right\rfloor = \lfloor t_i 2^{k-1} \rfloor + 2^k$, which is equivalent to

$$\lfloor t_i 2^{k-1} \rfloor + 2^k \le \sqrt{2}\left(\lfloor t_i 2^{k-2} \rfloor + \gamma_i 2^{k-l_i-2} + \frac{1}{2} \right) < \lfloor t_i 2^{k-1} \rfloor + 2^k + 1,$$

or in other words,

$$0 \le 2^{k-l_i-1}\left(\beta_i - \frac{\sqrt{2}}{2}\beta_i + \frac{\sqrt{2}}{2}\gamma_i - 2^{l_i+1} \right) + \sqrt{2}\,\lfloor \alpha_i \sqrt{2}\, 2^{k-l_i-2} \rfloor - \lfloor \alpha_i \sqrt{2}\, 2^{k-l_i-1} \rfloor + \frac{\sqrt{2}}{2} < 1.$$

Since $\gamma_i - \beta_i = 2\alpha_i$ and $\alpha_i + \beta_i = 2^{l_i+1}$ this is the same as

$$0 \le \{\alpha_i \sqrt{2}\, 2^{k-l_i-1}\} - \sqrt{2}\,\{\alpha_i \sqrt{2}\, 2^{k-l_i-2}\} + \frac{\sqrt{2}}{2} < 1. \tag{7}$$

Relation (7) is true since $0 \le \{x\} - \sqrt{2}\,\{x/2\} + \sqrt{2}/2 < 1$ for all $x \in \mathbb{R}$.

Now, assume relation (6). We have to ensure that $\lfloor \sqrt{2}\,(v_{2k+1} + \varepsilon) \rfloor = \lfloor t_i 2^{k-1} \rfloor + \gamma_i 2^{k-l_i-1}$, or equivalently,

$$\lfloor t_i 2^{k-1} \rfloor + \gamma_i 2^{k-l_i-1} \le \sqrt{2}\left(\lfloor t_i 2^{k-1} \rfloor + 2^k + \varepsilon\right) < \lfloor t_i 2^{k-1} \rfloor + \gamma_i 2^{k-l_i-1} + 1.$$

Here we end up with

$$0 \le (1 - \sqrt{2})\{\alpha_i \sqrt{2}\, 2^{k-l_i-1}\} + \sqrt{2}\,\varepsilon < 1, \tag{8}$$

which is true provided $1 - \sqrt{2}/2 \le \varepsilon < \sqrt{2}/2$. This interval includes all of the intervals $[\xi_{1,i}, \xi_{2,i})$ in Theorem 3.2, and hence there is no additional restriction on ε.

It remains to check the initial conditions. This task encompasses some straightforward calculations; we only give the main steps. First, we have to guarantee that (5) is true for $k = l_i + 2$. Of course, this crucially depends on the choice of ε. Since $v_n(\varepsilon)$ is non-decreasing for increasing values of ε, there is at most one semi-open real interval $[\bar{\xi}_{1,i}, \bar{\xi}_{2,i})$ for ε such that

$$v_{2(l_i+2)} = \lfloor t_i 2^{l_i} \rfloor + \gamma_i = \lfloor \alpha_i \sqrt{2} - \beta_i \rfloor + 2\alpha_i + \beta_i = \lfloor \alpha_i \sqrt{2} \rfloor + 2\alpha_i. \tag{9}$$

We will show that $[\bar{\xi}_{1,i}, \bar{\xi}_{2,i}) = [\xi_{1,i}, \xi_{2,i})$. It is not difficult to crank out a reasonable guess for $\bar{\xi}_{1,i}$ with the help of a computer. In fact, $v_{2(l_i+2)}$ is a piecewise constant function in ε with only a finite number of jump discontinuities. Thus, we can get a close approximation of $\bar{\xi}_{1,i}$ by interval halving. Furthermore, from Definition 3.1 we see that $\bar{\xi}_{1,i}$ (if it exists) has the form $\frac{c_i}{2}\sqrt{2} - d_i$ for some integers $c_i, d_i \in \mathbb{Z}$. We use *Maple 11* (`PolynomialTools[MinimalPolynomial]`) to calculate an approximate minimal polynomial of degree two with "small" coefficients to identify a conjectured value for $\bar{\xi}_{1,i}$. Again, we have to ensure that the value still satisfies (9).

As an illustration, let $i = 6$ and consider

$$v_{2(l_i+2)} = v_{62} = v_{62}(\varepsilon), \qquad \lfloor \alpha_i \sqrt{2} \rfloor + 2\alpha_i = 2749487923.$$

Figure 2 shows the location of the jumps in the graph of $v_{62}(\varepsilon)$ for $\varepsilon \in [0.40, 0.60]$. By the above procedure we find that $\bar{\xi}_{1,6}$ is "close" to

$$\xi_{1,6} = 1296121037\sqrt{2}/2 - 916495974 = 0.5012400\ldots.$$

Once more, we use *Maple* with the *ansatz* $\varepsilon = \xi_{1,i} - \delta$, where δ denotes a small positive quantity, to show that $\varepsilon = \xi_{1,i}$ is indeed the smallest value which satisfies (9). This is a symbolic computation and does not involve high-precision arithmetic. In a similar fashion, we show that $\bar{\xi}_{2,i} = \xi_{2,i}$. It is important to note that the values of $v_1, v_3, \ldots, v_{2(l_i+1)+1}$ remain unchanged for $\varepsilon \in [\xi_{1,i}, \xi_{2,i})$ for every fixed $i \in I$. Moreover, a routine calculation confirms that (6) is true for $0 \le k \le l_i + 1$.

Finally, we have to treat the case $i = 5$, which is less involved than the cases $i \in I$. Here we directly show that

$$v_{2k} = \lfloor t_i 2^{k-2} \rfloor + 2^{k-2} \qquad \text{and} \qquad v_{2k+1} = \lfloor t_i 2^{k-1} \rfloor + 2^k$$

for $k \ge 1$, so that we do not have to bother about initial conditions. (We leave the details to the interested reader.)

Summing up, we have that the intervals $[\xi_{i,1}, \xi_{i,2})$ are disjoint for $i = 1, 2, \ldots, 8$ and completely cover $[1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$. This finishes the proof of Theorem 3.2. $\square$

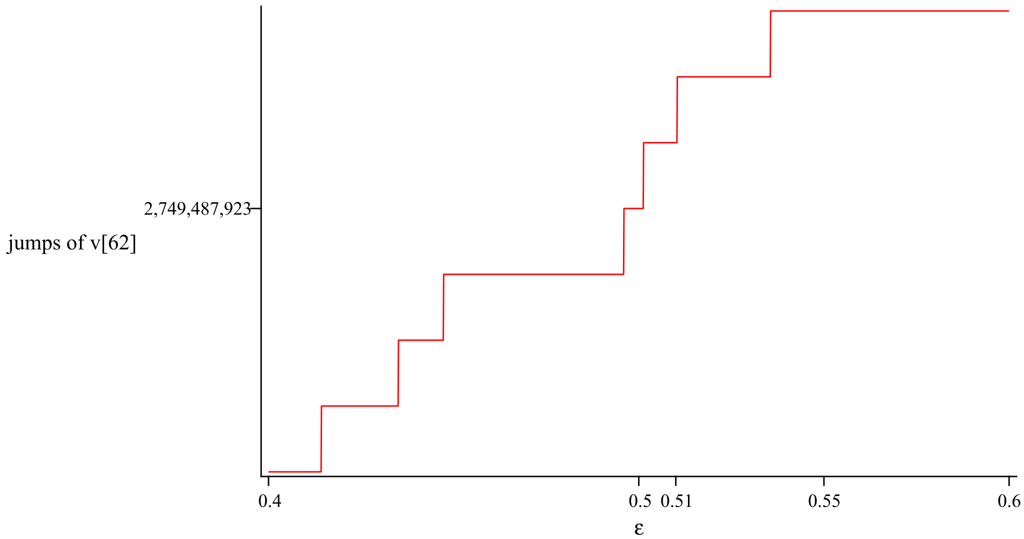

Figure 2: The jumps of $v_{62}(\varepsilon)$ for $i = 6$ and $0.4 \leq \varepsilon \leq 0.6$.

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